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THEORY OF COUPLED CIRCUITS, UNDER THE ACTION OF AN IMPRESSED ELECTROMOTIVE FORCE, WITH APPLICATIONS TO RADIOTELEGRAPHY.

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I. Introduction.

Persistent, or sustained, electric oscillations have recently come into extensive use in wireless telegraphy. With these oscillations, which are produced continuously while the transmitting key is depressed, tens of thousands of waves arrive at the receiving station during even the production of a single dot of the Morse code. This permits the establishment of a practically steady state at the receiving station, so that by the use of these persistent oscillations the mathematical treatment of the problem of the resonance conditions at the receiving station reduces to a problem of forced vibration.

The exact solution for the radiotelegraphic circuits, however, still presents considerable difficulty on account of the effect of the distributed capacity of the antennae. An approximation to a solution of the practical problem can be obtained by supposing that the antenna of the receiving station of the practical case can be replaced by a localized capacity so that the circuits become those represented in Figure 1. While this simplified system is a considerable departure from the actual practical circuits, calculations made from the simplified circuits seem nevertheless to be of importance, because the resonance in the simplified system is sharper than in the actual circuits, and the simplified computations thus afford a means of assigning certain theoretical limits to practical attainments.

It is the purpose of the present communication to give a solution of the equations representing the flow of electricity in a system of circuits of the form of Figure 1, under the action of a sinusoidal impressed electromotive force at e, and to make from this solution deductions in

regard to the wireless telegraphic receiving station. It will not be necessary to neglect the resistances of the system, and, in fact, the influence of the resistances upon the resonance relations and upon the resultant current is the most interesting part of the investigation. The results of the mathematical treatment are illustrated by numerical examples.

II. An Experiment on Received Current.

In pursuing the mathematical development I have received aid from an examination of some experimental data previously published, and I

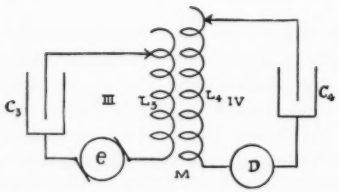

Figure 1. Diagram of coupled circuits with impressed e. m. f.

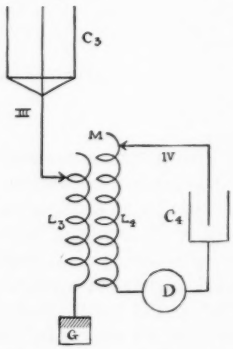

Figure 2. Electro-magnetically coupled wireless telegraph receiving station.

take the liberty of presenting one set of these experimental results [1] as an introduction to the theoretical investigation. These experimental data were obtained five years ago with a spark-discharge method of excitation instead of with a persistent source of waves, so that the experimental case is to be regarded in this connection merely as an aid to a concrete statement of the problem under consideration.

The experiments were made with a receiving station of the electromagnetically connected type, like that shown in Figure 2. With electric waves of fixed period arriving from a distant sending station the receiving station of Figure 2 was attuned by adjustment of the condenser C_4 which is in the side-circuit. Readings of the received current in the side-circuit were made by the use of a low-resistance high-frequency

[1] G. W. Pierce, Physical Review, **20**, 220, 1905.

dynamometer D. The inductances L_3 and L_4 and their mutual inductance were kept constant. Since, however, a single variation at the receiving circuit — namely, the variation of the condenser C_4 — is not sufficient to disclose the resonance conditions at the receiving station, the length of the receiving antenna was also given various values. The results are plotted in the curves of Figure 3, and were obtained as follows:

With a given length (23.8 meters) of the four-wire receiving antenna the capacity C_4 was set at a particular value and the deflection of the

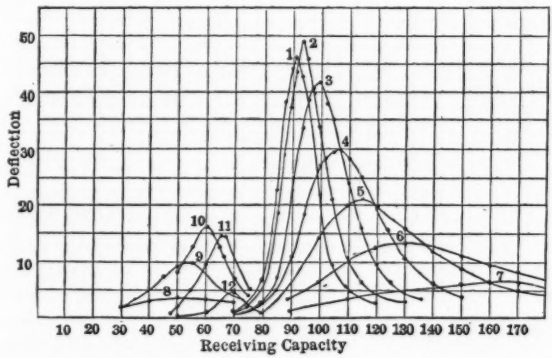

FIGURE 3. Family of experimental curves obtained with various heights of antenna (each height giving one of the curves), and various values of receiving capacity C_4.

dynamometer D was read. The capacity C_4 was then set at another value and the deflection of the dynamometer was read. Thus keeping the length of antenna at 23.8 meters, I took the readings of the dynamometer for a whole series of values of C_4. The results are shown in Curve 1 of Figure 3, in which the deflections of the dynamometer (current square) are plotted against values of "receiving capacity" C_4 (in arbitrary units). We shall call this kind of a curve a "resonance curve." When this resonance curve (Curve 1) was completed, the antenna was shortened to 20.8 meters and a second resonance curve, Curve 2, Figure 3, was taken. Again shortening the antenna successively to 17.8, 15.8, 14.8, 13.8, and 12.8 meters, I obtained the curves 3, 4, 5, 6, and 7 respectively of Figure 3.

From this it is seen that for each particular capacity of the antenna, varied by varying the length, there is a characteristic capacity C_4 that

gives a maximum of current. We shall call the condenser capacity that gives a maximum of current the "resonant C_4" for a given C_3. In seeking an expression for the condition for resonance, it is to be noticed that, as C_3 has been diminished, the value of the "receiving capacity" C_4 required for resonance has been increased (compare

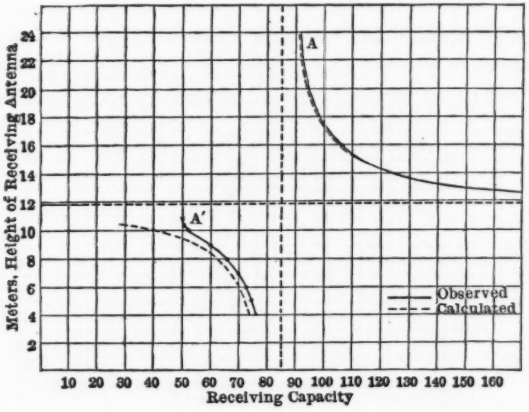

FIGURE 4. Relation of height of Antenna to C_4 for resonance.

Curves 1 to 7). On passing to a smaller length of antenna, in the neighborhood of 11.8 meters, the value of C_4 for resonance became very great and the amount of current received became inappreciably small. With this length of antenna, 11.8 meters, the condition of the receiving circuit was at its worst, for upon further decreasing the length of antenna to 10.5 meters resonance reappeared in the form of Curve 8. The resonance relation had undergone a discontinuity and the capacity C_4 for resonance had jumped back toward the origin. Now, decreasing the length of the antenna further to 10, 9, 8, and 7 meters successively, I obtained the resonance curves 9, 10, 11, and 12 respectively[2] of Figure 3.

[2] Curves 9 to 12 are plotted magnified about five times in comparison with Curves 1 to 8. In examining these curves it should be borne in mind that the decrease of length or height of the receiving antenna carries with it a double effect; namely, (1) a decrease of *reach* of the antenna into the field of force, and (2) a shift of the resonance relations of the receiving station with respect to the incident waves. We are at present concerned only with the second of these effects.

The relation between the height of antenna, H, and the capacity C_4 required for resonance was found to be approximately represented by the empirical equation:

$$(H - 11.8)(C_4 - 84.6) = 88, \tag{1}$$

which is the equation of an equilateral hyperbola with axes at $H = 11.8$ and $C_4 = 84.6$. The nature of the agreement between the observed and the calculated values is shown in Figure 4. Evidently the relation expressed in equation (1), though an interesting approximation, is not exact.

III. Theoretical Treatment.

Let us now turn from the experiment to the theory of the oscillation. The problem undertaken is the investigation of the relative current in the detector circuit (Circuit IV of Figure 1) for various adjustments of the constants of Circuit III and Circuit IV. In the theoretical treatment this carries with it (1) a determination of the adjustments that must be made to obtain resonance, (2) a determination of the adjustment for best resonance, (3) a determination of the effect of the resistances on the resonance relations and on the amount of current receivable, (4) a discussion of the resistance that a detector must have for greatest sensitiveness, (5) a computation of the amount of disturbing current that will be obtained from an undesired source of waves, and (6) a quantitative judgment as to the sharpest selectivity that can be attained by circuits of the form of Figure 2.

As stated in the Introduction, in treating these general questions it has been found necessary to depart from strict observance of the actual practical wireless-telegraphic conditions and to assume the capacity of the antenna, which is a distributed capacity in practical wireless telegraphy, to be replaceable by a localized capacity C_3. This modification of the problem will not completely destroy the validity of the discussion, because the simplified problem enables us to derive certain important conclusions in regard to the problem with the less simple conditions.

Referring to the localized-capacity circuits of Figure 1, and supposing an electromotive force $E\cos\omega t$ at e, the differential equations of the current in the two circuits are

$$L_3\frac{\partial x}{\partial t} + M\frac{\partial y}{\partial t} + R_3 x + \frac{1}{C_3}\int x\,dt = E\cos\omega t, \tag{2}$$

$$L_4\frac{\partial y}{\partial t} + M\frac{\partial x}{\partial t} + R_4 y + \frac{1}{C_4}\int y\,dt = 0, \tag{3}$$

in which x and y are the values of the current in the Circuits III and IV respectively. Eliminating x between these equations we have

$$(4)\quad (L_3L_4 - M^2)\frac{\partial^4 y}{\partial t^4} + (R_4L_3 + R_3L_4)\frac{\partial^3 y}{\partial t^3} + \left(\frac{L_3}{C_4} + \frac{L_4}{C_3} + R_3R_4\right)\frac{\partial^2 y}{\partial t^2} + \left(\frac{R_3}{C_4} + \frac{R_4}{C_3}\right)\frac{\partial y}{\partial t} + \frac{y}{C_3C_4} = -EM\omega^3 \sin \omega t.$$

If for brevity we call the left-hand member of equation (4) $f(y)$, the complete solution of (4) is any particular solution of (4) plus the general solution of

$$(5)\quad f(y) = 0.$$

Now the general solution of (5) involves exponentials with negative exponents as multipliers, and becomes zero after a few oscillations, so that all we need for the current y in case a large number of oscillations are performed, as with a persistent source of waves, is the "steady-state" solution for equation (4).

In order to get the steady-state solution of (4) let us write, in the place of (4), the equation

$$(6)\quad (L_3L_4 - M^2)\frac{\partial^4 y}{\partial t^4} + (R_3L_4 + R_4L_3)\frac{\partial^3 y}{\partial t^3} + \left(\frac{L_3}{C_4} + \frac{L_4}{C_3} + R_3R_4\right)\frac{\partial^2 y}{\partial t^2} + \left(\frac{R_3}{C_4} + \frac{R_4}{C_3}\right)\frac{\partial y}{\partial t} + \frac{y}{C_3C_4} = -EM\omega^3 e^{i\omega t},$$

which is (4) with $\sin \omega t$ replaced by an appropriate exponential with imaginary exponent.

Our required solution for (4) can then be obtained from a solution of (6) by getting y from (6), rationalizing the result, taking the imaginary part and dividing by i.

Now a particular solution of (6) is seen to be of the form of

$$(7)\quad y = Ye^{i\omega t},$$

in which Y is to be determined by substitution of (7) in equation (6).

Making this substitution we obtain

$$(8)\quad AY\omega^4 - BiY\omega^3 - CY\omega^2 + DiY\omega + FY = -EM\omega^3,$$

where A, B, C, D, and F are the coefficients of equation (6). Whence

$$(9)\quad Y = \frac{-EM\omega^3}{(A\omega^4 - C\omega^2 + F) - (B\omega^3 - D\omega)i},$$

or writing the real part of the denominator as P, the imaginary as Qi,

$$Y = \frac{-EM\omega^3}{P - Qi}; \tag{10}$$

whence from equation (7),

$$y = \frac{-EM\omega^3 e^{i\omega t}}{P - Qi}, \tag{11}$$

$$= \frac{-EM\omega^3 (\cos\omega t + i \sin\omega t).}{P - Qi}$$

Rationalizing equation (11), taking only the imaginary part and dividing by i, we have

$$y = \frac{-EM\omega^3 (P \sin\omega t + Q\cos\omega t),}{P^2 + Q^2}$$

$$= \frac{-EM\omega^3 \sin\left(\omega t + \tan^{-1}\frac{Q}{P}\right)}{\sqrt{P^2 + Q^2}}. \tag{12}$$

Replacing P and Q in the denominator by their values in terms of the constants of the circuits, we have

$$y = \frac{-EM\omega^3 \sin\left(\omega t + \tan^{-1}\frac{Q}{P}\right)}{\sqrt{\left\{(L_3L_4 - M^2)\omega^4 - \left(\frac{L_3}{C_4} + \frac{L_4}{C_3} + R_3R_4\right)\omega^2 + \frac{1}{C_3C_4}\right\}^2 + \left\{(R_4L_3 + R_3L_4)\omega^3 - \left(\frac{R_4}{C_3} + \frac{R_3}{C_4}\right)\omega\right\}^2}} \tag{13}$$

Equation (13) is a well-known solution of equation (4), and gives the value of the current y in Circuit IV after the effect of the free-period initial disturbance has subsided. It is seen that the current y in Circuit IV is sinusoidal, with the frequency of the e.m.f. impressed on Circuit III. We shall concern ourselves only with the absolute value of the amplitude Y of this current.

Dividing numerator and denominator of equation (13) by ω^2 and factoring, we obtain

$$Y = \frac{EM\omega}{\sqrt{\left\{\left(L_3\omega - \frac{1}{C_3\omega}\right)\left(L_4\omega - \frac{1}{C_4\omega}\right) - M^2\omega^2 - R_3R_4\right\}^2 + \left\{R_4\left(L_3\omega - \frac{1}{C_3\omega}\right) + R_3\left(L_4\omega - \frac{1}{C_4\omega}\right)^2\right\}}} \tag{14}$$

Now let

$$U = L_3\omega - \frac{1}{C_3\omega}, \qquad V = L_4\omega - \frac{1}{C_4\omega}. \tag{15}$$

Then equation (14) may be written

$$Y = \frac{EM\omega}{\sqrt{(UV - M^2\omega^2 - R_3R_4)^2 + (R_4U + R_3V)^2}}. \tag{16}$$

This is another form of the expression for the current in Circuit IV, after the effect of the free-period initial disturbance has subsided. From equation (16) it can be seen that L_3 and C_3 enter in the equation only in the form U; L_4 and C_4 enter only in the form V.

In seeking the dependence of Y on the constants of the circuits let us now follow mathematically the steps taken in the experiment above described.

First, we shall take a fixed value of U and determine what value of V makes Y a maximum. This value of V that makes Y a maximum we shall call the "resonant value of V." The resonant value of V is obtained by making

$$\frac{\partial Y}{\partial V} = 0. \tag{17}$$

This gives

$$(UV - M^2\omega^2 - R_3R_4)\,U + (R_4U + R_3V)\,R_3 = 0 \tag{18}$$

or

$$V = \frac{M^2\omega^2 U}{U^2 + R_3^2} \tag{19}$$

Equation (19) gives the value of V for resonance with any particular given value of U. This relation will be further examined in a later section.

By the use of equation (19), when we have a given fixed value of U we can calculate the value of V for resonance. The current in Circuit IV for this resonant value of V may be obtained by substituting the value of V given by equation (19) into the equation for Y (equation (16)). If we designate this resonant value of Y by Y_{max}, we have

$$Y_{max} = \frac{EM\omega}{\sqrt{\left(\frac{M^2\omega^2U^2}{U^2 + R_3^2} - M^2\omega^2 - R_3R_4\right)^2 + \left(R_4U + \frac{R_3M^2\omega^2U}{U + R_3^2}\right)^2}}$$

$$= \frac{EM\omega}{\sqrt{\left(\frac{R_3^2M^2\omega^2}{U^2 + R_3^2} + R_3R_4\right)^2 + \left(\frac{R_3M^2\omega^2U}{U^2 + R_3^2} + R_4U\right)^2}}$$

Factoring the denominator of this expression, we have

$$Y_{max} = \frac{EM\omega}{\left(\frac{R_3M^2\omega^2}{U^2 + R_3^2} + R_4\right)\sqrt{R_3^2 + U^2}}. \tag{20}$$

For any given value of U and the necessary other constants of the circuits, equation (19) enables us to compute the resonant adjustment of Circuit IV, and equation (20) gives the amplitude of the received current at the resonant adjustment. We have thus found the best V and the best Y *for a given* U. It is proposed next to find what would be the best value to give to U, while also keeping V at its best value, and thus to determine the best possible Y, which we shall call $Y_{max\,max}$.

To obtain the best U we must apply to equation (20) the condition

$$\frac{\partial Y_{max}}{\partial U} = 0.$$

This gives

$$-\frac{R_3 M^2\omega^2 \cdot 2\,U}{(U^2 + R_3^2)^2}\sqrt{R_3^2 + U^2} + \left(\frac{R_3 M^2\omega^2}{U^2 + R_3^2} + R_4\right)\frac{U}{\sqrt{R_3^2 + U^2}} = 0,$$

or

$$-R_3 M^2\omega^2 U + R_4 U^3 + R_3^2 R_4 U = 0. \tag{20 a}$$

Whence, omitting for the present the case of $U = 0$, which is treated on page 302, we have, after dividing by U and transposing,

$$U_{opt} = \pm\sqrt{\frac{R_3}{R_4}(M^2\omega^2 - R_3 R_4)}, \tag{21}$$

in which the subscript "opt" is introduced to designate the optimum value. At the same time V must satisfy equation (19) which combined with equation (21) gives

$$V_{opt} = \pm\sqrt{\frac{R_4}{R_3}(M^2\omega^2 - R_3 R_4)}. \tag{22}$$

According to the conditions imposed by equation (19) U_{opt} and V_{opt} must either both be positive or both be negative. They cannot have opposite signs.

Equations (21) and (22) show what values to give U and V in order to obtain the largest possible current (which we shall call $Y_{max\,max}$) in Circuit IV. The value of the current, under these conditions, is found by substituting the optimum value of U, namely U_{opt} of equation (21), into the equation for Y_{max} (equation 20). When this is done, we have, after simplification,

$$Y_{max\,max} = \frac{E}{2\sqrt{R_3 R_4}}. \tag{23}$$

Equation (23) gives the max max current in Circuit IV provided equations (21) and (22) express the optimum resonance relation. Instead of (21) an alternative possible solution of equation (20_a) is

$$U_{opt} = 0.$$

Under this condition, according to equation (19)

$$V_{opt} = 0, \text{ also.}$$

This would give by equation (20)

$$(Y_{max\ max})_0 = \frac{EM\omega}{M^2\omega^2 + R_3R_4}. \tag{23 a}$$

The question arises under what conditions $(Y_{max\ max})_0$ of equation (23 a) is greater than $Y_{max\ max}$ of equation (23). The answer is seen to be that

$$(Y_{max\ max})_0 > Y_{max\ max}$$

when

$$\frac{EM\omega}{M^2\omega^2 + R_3R_4} > \frac{E}{2\sqrt{R_3R_4}};$$

that is, when

$$M\omega + \frac{R_3R_4}{M\omega} < 2\sqrt{R_3R_4}.$$

Squaring,

$$M^2\omega^2 + 2\,R_3R_4 + \frac{R_3^2R_4^2}{M^2\omega^2} < 4\,R_3R_4,$$

i. e.,

$$M^2\omega^2 - 2\,R_3R_4 + \frac{R_3R_4}{M^2\omega^2} < 0.$$

Extracting square root,

$$M\omega - \frac{R_3R_4}{M\omega} < 0,$$

$$M^2\omega^2 - R_3R_4 < 0. \tag{23 b}$$

In this case, the conditions (21) and (22) would give imaginary value of U_{opt} and V_{opt}.

Whence we conclude, that we are to use equations (21), (22) and (23) as solutions of the resonance problems, whenever $M^2\omega^2 > R_3R_3$. In other cases the alternative values

$$U_{opt} = 0, \qquad V_{opt} = 0, \quad \text{and} \quad (Y_{max\ max})_0 = \frac{EM\omega}{M^2\omega^2 + R_3R_4}$$

are to be used.

IV. Numerical Examination of the General Resonance Relation.

The equations above derived we shall now submit to numerical examination. Let us first examine the general resonance relation as expressed in equation (19). Replacing U and V in equation (19) by

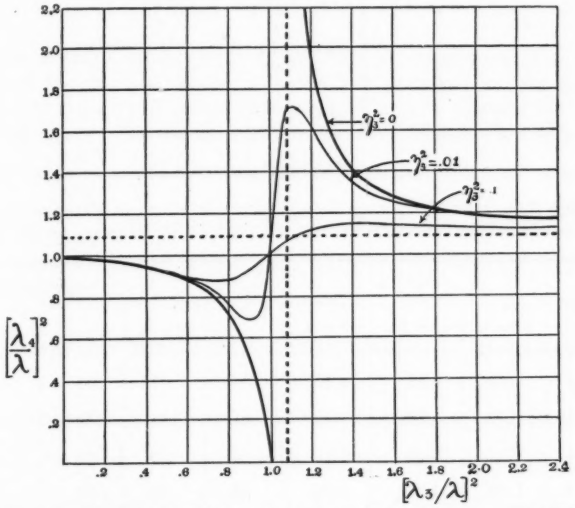

FIGURE 5. Theoretical relation of λ_4 to λ_3 for resonance. The different curves are for different values of damping in Circuit III. Given $\tau = .29$.

their values (15), we have the general resonance relation in the following form:

$$L_4\omega - \frac{1}{C_4\omega} = \frac{M^2\omega^2\left(L_3\omega - \frac{1}{C_3\omega}\right)}{\left(L_3\omega - \frac{1}{C_3\omega}\right)^2 + R_3^2}, \tag{24}$$

or

$$1 - \left(\frac{\omega_4}{\omega}\right)^2 = \frac{\tau^2\left\{1 - \left(\frac{\omega_3}{\omega}\right)^2\right\}}{\left\{1 - \left(\frac{\omega_3}{\omega}\right)^2\right\}^2 + \eta_3^2}, \tag{25}$$

in which

$$(26)\qquad \begin{cases} \omega_4^2 = \dfrac{1}{L_4 C_4}, \\ \omega_3^2 = \dfrac{1}{L_3 C_3}, \\ \tau^2 = \dfrac{M^2}{L_3 L_4}, \\ \eta_3 = \dfrac{R_3}{L_3 \omega}, \\ \eta_4 = \dfrac{R_4}{L_4 \omega}. \end{cases}$$

TABLE I.

SHOWING RESONANT VALUES OF $(\omega/\omega_4)^2$ CORRESPONDING TO VARIOUS VALUES OF $(\omega/\omega_3)^2$. THE COEFFICIENT OF COUPLING, τ, IS ASSUMED TO BE .29.

$(\omega/\omega_3)^2$.	$(\omega/\omega_4)^2$ for.			$(\omega/\omega_3)^2$.	$(\omega/\omega_4)^2$ for.		
	$\eta_3^2 = 0.$	$\eta_3^2 = .01.$	$\eta_3^2 = .1.$		$\eta_3^2 = 0.$	$\eta_3^2 = .01.$	$\eta_3^2 = .1.$
2.50	1.16	1.17	1.12	.95	.373	.75	.96
2.00	1.20	1.19	1.13	.90	.54	.70	.93
1.66	1.27	1.25	1.15	.82	.70	.75	.89
1.43	1.39	1.33	1.15	.77	.78	.80	.88
1.25	1.72	1.51	1.13	.71	.82	.83	.88
1.17	2.27	....	....	.67	.85	.86	.89
1.11	6.25	1.72	1.08	.50	.93	.93	.93
1.05	−1.47	1.50	1.04	.30	.96	.96	.96
1.00	0.00	0.00	0.00	.25	.97	.97	.97

Some numerical calculations are given in Table I and plotted in Figure 5. Assuming $\tau = .29$, three different values of η_3^2 have been employed in calculating Table I, by the aid of equation (25) namely, $\eta_3^2 = 0$, .01, and .1, respectively.

Table I and Figure 5 give the resonant values of $\left(\dfrac{\omega}{\omega_4}\right)^2$ correspond-

ing to different values of $\left(\frac{\omega}{\omega_3}\right)^2$. In examining the table and the figure it should be remembered that

$$(27) \qquad \begin{cases} \left(\frac{\omega}{\omega_4}\right)^2 = \left(\frac{\lambda_4}{\lambda}\right)^2 = L_4 C_4 \omega^2, \\ \left(\frac{\omega}{\omega_3}\right)^2 = \left(\frac{\lambda_3}{\lambda}\right)^2 = L_3 C_3 \omega^2, \end{cases}$$

in which ω and λ are respectively the angular velocity and the wave length of the incident waves; λ_3 and λ_4 are the wave lengths corresponding to the natural period of Circuits III and IV respectively when standing alone. The abscissas of the curves of Figure 5 are values of $\left(\frac{\lambda_3}{\lambda}\right)^2$, or $\left(\frac{\omega}{\omega_3}\right)^2$, which equals $L_3 C_3 \omega^2$; the ordinates are values of $\left(\frac{\lambda_4}{\lambda}\right)^2$, or $\left(\frac{\omega}{\omega_4}\right)^2$, which equals $L_4 C_4 \omega^2$. For fixed values of L_3, L_4, and ω the abscissas and ordinates are, therefore, proportional to C_3 and C_4 respectively.

V. The General Resonance Relation (*continued*). Special Case Where $\eta_3^2 = 0$.

An examination of equation (25) shows that when $\eta_3^2 = 0$ the curve of $\left(\frac{\lambda_3}{\lambda}\right)^2$ vs. $\left(\frac{\lambda_3}{\lambda}\right)^2$ for resonance is an equilateral hyperbola, with horizontal asymptote at $\left(\frac{\lambda_4}{\lambda}\right)^2 = \frac{1}{1-\tau^2}$, and vertical asymptote at $\left(\frac{\lambda_3}{\lambda}\right)^2 = \frac{1}{1-\tau^2}$. This curve with its asymptotes is also plotted in Figure 5. In the part of the curves plotted in Figure 5, even when η_3^2 is not equal to zero the curve does not appreciably depart from the equilateral hyperbola provided $\eta_3^2 \lessgtr .001$. The corresponding curve in C_4 vs. C_3 for $\eta_3^2 = 0$, or $\eta_3^2 \lessgtr .001$, has its asymptotes at $C_3 = \frac{1}{(1-\tau^2) L_3 \omega^2}$ and $C_4 = \frac{1}{(1-\tau^2) L_4 \omega^2}$, and the equation giving the resonance relation of C_4 to C_3 in this case, obtained by a transformation of equation (25), is

$$(28) \qquad \left(C_3 - \frac{1}{(1-\tau^2) L_3 \omega^2}\right)\left(C_4 - \frac{1}{(1-\tau^2) L_4 \omega^2}\right) = \frac{\tau^2}{(1-\tau^2)^2 L_3 L_4 \omega^4}.$$

This is the equation to which the empirical equation (1) corresponds, provided the height of antenna H is proportional to the capacity of the antenna C_3, assumed localized. The condition that C_3 should be localized, and particularly the conditions that the incident waves should be undamped, persistent and single-valued, were not fulfilled in the illustrative experiment cited above, and therefore it does not seem important to enter into a detailed comparison of equation (28) with equation (1). This has been partially done in the previous account of the experiment.[3]

The manner in which the resonance relation is affected by the resistance of Circuit III, corresponding to the antenna circuit, is shown in the curves marked $\eta_3^2 = .01$ and $\eta_3^2 = .1$ of Figure 5. The resistance R_4 of Circuit IV, corresponding to the detector circuit, is without effect in determining the form of these curves, which represent the general resonance relation. On the other hand, R_4 does have an effect in determining at which point of the curve of resonance relation the resonance is best, and R_4 is also significant in determining the sharpness of resonance. Some computations on this subject are given below.

VI. On the Optimum Resonance Relation.

Let us next examine the conditions for best resonance. We are still concerned merely with the steady-state vibration of the coupled circuits of Figure 1, under the action of the impressed sinusoidal electromotive force. The conditions for best resonance provided $M^2\omega^2 > R_3R_4$ are given in the equations (21) and (22), which after substitution from (15) become respectively

$$\left(L_3\omega - \frac{1}{C_3\omega}\right)_{opt} = \pm \sqrt{\frac{R_3}{R_4}(M^2\omega^2 - R_3R_4)}, \tag{29}$$

and

$$\left(L_4\omega - \frac{1}{C_4\omega}\right)_{opt} = \pm \sqrt{\frac{R_4}{R_3}(M^2\omega^2 - R_3R_4)}. \tag{30}$$

Dividing both sides of equation (29) by $L_3\omega$ and employing the notation of equation (26) we have

$$1 - \left(\frac{\omega_3}{\omega}\right)^2_{opt} = \pm \eta_3 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}. \tag{31}$$

[3] G. W. Pierce, Physical Review, **20**, 220, 1905.

In like manner equation (30) becomes

$$1 - \left(\frac{\omega_4}{\omega}\right)^2_{opt} = \pm\, \eta_4 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}. \tag{32}$$

Replacing the ratio of angular velocities by the reciprocal ratio of wave lengths, in accordance with equations (27), we may transform equations (31) and (32) into the following equations:

$$\left(\frac{\lambda_3}{\lambda}\right)^2_{opt} = \frac{1}{1 \pm \eta_3 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}}, \tag{33}$$

and

$$\left(\frac{\lambda_4}{\lambda}\right)^2_{opt} = \frac{1}{1 \pm \eta_4 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}}. \tag{34}$$

It should be borne in mind that in order to get properly corresponding resonant values of λ_3 and λ_4, one must use either the plus sign in both of the equations or else the minus sign in both the equations. If one employs the plus sign in one of the equations and the minus sign in the other equation, the values of λ_3 and λ_4 so obtained are not appropriate simultaneous adjustments for best resonance. This is seen by an examination of equation (19).

By an examination of the discussion on p. 302 it will be seen that the optimum condition in the form of equations (33) and (34) can be attained only provided

$$\frac{\tau^2}{\eta_3\eta_4} > 1. \tag{35}$$

In order to facilitate the computation of λ_3 and λ_4 for various values of τ, η_3, η_4 and λ, equations (33) and (34) may be put in the form

$$\left(\frac{\lambda_3}{\lambda}\right)_{opt} = \frac{1}{\sqrt{1 \pm \phi_3}}, \quad \text{where } \phi_3 = \eta_3 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}, \tag{36}$$

and

$$\left(\frac{\lambda_4}{\lambda}\right)_{opt} = \frac{1}{\sqrt{1 \pm \phi_4}}, \quad \text{where } \phi_4 = \eta_4 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}. \tag{37}$$

The following table (Table II) gives the values of $\left(\frac{\lambda_3}{\lambda}\right)_{opt}$ for various values of ϕ_3.

TABLE II.

VALUES OF $\left(\frac{\lambda_3}{\lambda}\right)_{opt.}$ CORRESPONDING TO DIFFERENT VALUES OF ϕ_3. COMPUTED FROM EQUATIONS (36) AND (37).

ϕ_3.	$\left(\frac{\lambda_3}{\lambda}\right)_{opt.}$		ϕ_3.	$\left(\frac{\lambda_3}{\lambda}\right)_{opt.}$	
	using + sign.	using − sign.		using + sign.	using − sign.
.0	1.000	1.000	1.4	.646	Imaginary
.1	.953	1.054	1.5	.632	"
.2	.913	1.118	1.6	.620	"
.3	.877	1.196	1.7	.608	"
.4	.845	1.292	1.8	.597	"
.5	.817	1.414	1.9	.587	"
.6	.791	1.581	2.0	.577	"
.7	.767	1.825	2.1	.568	"
.8	.746	2.236	2.2	.559	"
.9	.725	3.162	2.3	.550	"
1.0	.707	∞	2.4	.542	"
1.1	.690	Imaginary	2.5	.535	"
1.2	.675	"	2.6	.527	"
1.3	.660	"			

These results are plotted in the curves of Figure 6, with $\left(\frac{\lambda_3}{\lambda}\right)_{opt}$ as ordinates and ϕ_3 as abscissas. The same curves give the values of $\left(\frac{\lambda_4}{\lambda}\right)_{opt}$ as ordinates provided the abscissas are read as values of ϕ_4. If the lower branch of the curve is used in obtaining the values of $\left(\frac{\lambda_3}{\lambda}\right)_{opt}$, the same branch must be employed in finding the corresponding values of $\left(\frac{\lambda_4}{\lambda}\right)_{opt}$. In like manner the top branch of the curve

may be employed in both cases, provided the value of ϕ_3 or ϕ_4 is less than unity. For values equal to unity the wave length for the top branch of the curve is infinite, and for greater values the wave lengths of the top branch become imaginary. In this case the optimum wave-length adjustment becomes single-values and must be read from the lower branch of the curve.

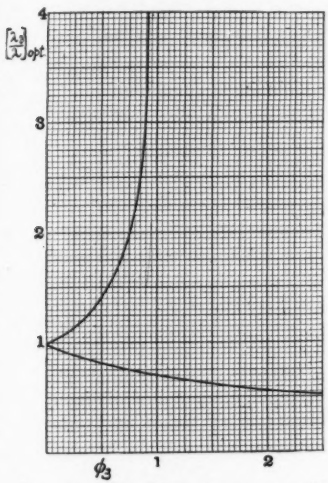

FIGURE 6. Auxiliary curve to assist in calculation of the optimum resonance adjustment. $\phi_3 = \eta_3 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}$. This curve also gives optimum value of λ_4 if ϕ_3 is replaced by $\phi_4 = \eta_4 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}$.

As an example of the manner of using the auxiliary curve (Figure 6) in the actual calculation of the optimum values of λ_3 and λ_4, let us take a special case. Suppose $\tau = .30$, $\eta_3 = .1$, let us give various values to η_4, and compute the corresponding optimum wave-length adjustments of the Circuits III and IV. A tabulation of the computation follows as Table III.

In compiling this table the values of ϕ_3 and ϕ_4 corresponding to the different values of η_4 were calculated by equations (36) and (37). The corresponding wave-length ratios were then taken from the curve of Figure 6.

The results contained in Table III are plotted in Figure 7. In a similar way the resonance relations for various values of τ and of η_3 may also be obtained, but the single example here computed and plotted serves to show the manner in which the coefficient of coupling and the damping factors contribute to determine the optimum resonance adjustment of the two circuits. The important facts to be noted are the following:

1. With given values of the coefficient of coupling and the damping factors of the two circuits the adjustment for best resonance is in general double valued. One may in general get best resonance either by setting both circuits to a wave length appropriately longer than that of the incident waves, or by setting both circuits to a wave length appropriately shorter than the incident waves.

TABLE III.

COMPUTATION OF OPTIMUM RESONANCE VALUES IN A SPECIAL CASE, IN WHICH
$\tau = .30$
$\eta_3 = .1$
η_4 IS GIVEN VARIOUS VALUES

η_4	ϕ_3	ϕ_4	$\left(\frac{\lambda_3}{\lambda}\right)_{opt.}$	$\left(\frac{\lambda_4}{\lambda}\right)_{opt.}$	$\left(\frac{\lambda_3}{\lambda}\right)_{opt.}$	$\left(\frac{\lambda_4}{\lambda}\right)_{opt.}$
.01	.943	.094	.720	.955	4.07	1.053
.02	.662	.133	.777	.940	1.73	1.070
.03	.539	.162	.806	.930	1.47	1.090
.04	.464	.186	.827	.917	1.37	1.110
.05	.412	.206	.840	.910	1.30	1.123
.06	.374	.224	.850	.905	1.26	1.135
.07	.345	.242	.862	.900	1.23	1.148
.08	.319	.255	.870	.895	1.205	1.152
.09	.300	.270	.875	.890	1.196	1.168
.1	.282	.282	.882	.882	1.180	1.180
.2	.187	.374	.917	.853	1.110	1.262
.3	.141	.423	.935	.837	1.078	1.310
.4	.112	.448	.950	.832	1.060	1.345
.5	.089	.445	.960	.830	1.048	1.340
.6	.071	.426	.970	.837	1.036	1.320
.7	.054	.378	.975	.852	1.032	1.262
.8	.035	.280	.980	.885	1.010	1.175
.9	.000	.000	1.000	1.000	1.000	1.000

Either pair of values under the brace is to be employed simultaneously for best resonance.

2. The adjustment for best resonance is materially influenced by the resistances of the two circuits. If, for example, with fixed incident

waves one tunes a wireless telegraph system of the coupled type to resonance with the use of a given detector, and then changes to a detector of different resistance, it is necessary to shift the wave length of both of the circuits in order to bring the system back to its best adjustment. This is a familiar experience, and serves to explain the

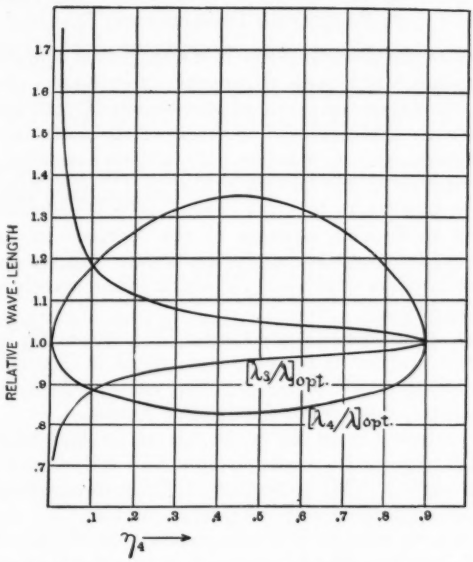

FIGURE 7. Relation of optimum wave-length adjustment to damping in Circuit IV, for a given value of η_3 and τ. ($\eta_3 = .1$, $\tau = .30$).

fact that with a vacuum detector of the type of Fleming's valve or Deforest's Audion or Hewitt's Mercury-arc detector one may attune the receiving station without change of its capacity or inductance merely by bringing a magnet up near the gaseous path of the detector. This merely changes the resistance of the circuit, and when the change is made in the proper direction and to the proper extent the system is thereby brought into resonance. The same thing may be effected in some cases by changing the strength of the local heating or ionizing current employed with detectors of this type.

3. The shifting of the resonance relation with change of resistance of the circuit has an influence on the sharpness of resonance when one

employs any of the ordinary detectors, because the resistance of the detectors is a function of the received current. This is a complex phenomenon and will enter into the consideration of sharpness of resonance, which is discussed below.

VII. On the Sharpness of Resonance and on the Possibility of Preventing Interference.

By reference to the previous pages it is seen that in order to obtain the best resonance, which gives a maximum of current in Circuit IV, it is necessary to adjust the period of both Circuit III and Circuit IV. We have given an expression for the value of the maximum current (that at best resonance) in equation (23), and we have also obtained a general expression for the current in Circuit IV (equation (16)), so that it is now possible to plot the current as best resonance is approached, and to form an estimate of the sharpness of resonance, whenever the constants of the circuits are known. This may conveniently be done in either of two ways, — which I shall classify as Case I and Case II, as follows:

Case I. Let us assume that the Circuit III is put at its best value (equation (21)), and let us compute the current in Circuit IV as the wave length of Circuit IV is varied. This corresponds to fixing the antenna circuit and tuning with the detector circuit.

Case II. Assume Circuit IV to be set at its best value (equation (22)) and compute the current in Circuit IV as the wave length of Circuit III is varied. This corresponds to fixing the constants of the detector circuit and tuning with the antenna circuit.

In either case we must know certain constants of the circuits, and I shall carry through the computation for both cases with several sets of constants. First it is necessary to transform the equations into suitable forms for making the computations.

Development of Equations for computing Case I. — The general expression for the amplitude of current in Circuit IV is given in equation (16); namely,

$$(16) \qquad Y = \frac{EM\omega}{\sqrt{(UV - M^2\omega^2 - R_3R_4)^2 + (R_4U + R_3V)^2}}.$$

Let us combine with this the condition that U shall have its optimum value, equation (22),

$$(22) \qquad U_{opt} = \pm \sqrt{\frac{R_3}{R_4}(M^2\omega^2 - R_3R_4)},$$

where $M^2\omega^2 > R_3R_4$, and let us suppose that V has, in general, not its optimum value, but a value k times its optimum value; that is,

$$V = k \cdot V_{opt}, \tag{38}$$

in which k is a variable parameter, which may be positive, negative, whole, or fractional.

Equation (16) then becomes

$$Y_{(U_{opt})} = \frac{EM\omega}{\sqrt{(k \cdot U_{opt} V_{opt} - M^2\omega^2 - R_3R_4)^2 + (R_4 U_{opt} + R_3 \cdot k \cdot V_{opt})^2}}. \tag{39}$$

Replacing U_{opt} and V_{opt} by their values (equations (21) and (22)) we have after simplification

$$Y_{(U_{opt})} = \frac{E}{\sqrt{(k-1)^2 (M^2\omega^2 - R_3R_4) + 4 R_3R_4}}. \tag{40}$$

Now dividing the square of equation (40) by the square of equation (23) we have

$$\left(\frac{Y_{(U_{opt})}}{Y_{max\ max}}\right)^2 = \frac{1}{\frac{(k-1)^2}{4}\left(\frac{M^2\omega^2}{R_3R_4} - 1\right) + 1} = \frac{1}{\frac{(k-1)^2}{4}\left(\frac{\tau^2}{\eta_3\eta_4} - 1\right) + 1}. \tag{41}$$

This equation gives the current in terms of the parameter k provided $\tau^2 > \eta_3\eta_4$. Let us now obtain the wave lengths in terms of the same parameter. The values of V and V_{opt} from equations (15) and (22) substituted in (38) give

$$L_4\omega - \frac{1}{C_4\omega} = \pm\, k \sqrt{\frac{R_4}{R_3}(M^2\omega^2 - R_3R_4)}. \tag{42}$$

Dividing (42) by $L_4\omega$ gives

$$1 - \left(\frac{\omega_4}{\omega}\right)^2 = \pm\, k \sqrt{\frac{R_4}{R_3}(M^2\omega^2 - R_3R_4)} = \pm\, k\eta_4 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}. \tag{43}$$

Whence

$$(44)\qquad \left(\frac{\lambda_4}{\lambda}\right)^2 = \frac{1}{1 \pm k\eta_4\sqrt{\dfrac{\tau^2}{\eta_3\eta_4} - 1}}$$

$$= \frac{1}{1 \pm k\phi_4}, \quad \text{where} \quad \phi_4 = \eta_4\sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}.$$

This equation (44) gives the relation of the wave lengths to the parameter k, and is applicable only provided U has its optimum value and $\tau^2 > \eta_3\eta_4$. The condition that V have its optimum value is conveniently expressed in the form of equation (36) above. Equations (44) and (36) must both be used with the same sign in order to be simultaneously correct.

In case $\tau^2 \gtreqless \eta_3\eta_4$, equations (44) and (41) cannot be employed. In this case U_{opt} and V_{opt} are both zero (see page 302), and a special investigation is necessary. This proves to be simple. Let us take equation (16), make $U = U_{opt} = 0$, and we have

$$(45)\qquad Y_{(U_{opt})} = \frac{EM\omega}{\sqrt{(M^2\omega^2 + R_3R_4)^2 + R_3{}^2V^2}}.$$

Expressing this result in terms of $Y_{max\ max}$ by dividing equation (45) by equation (23), we have

$$(46)\qquad \left(\frac{Y_{(U_{opt})}}{Y_{max\ max}}\right) = \frac{M^2\omega^2 \cdot 4\,R_3R_4}{(M^2\,\omega^2 + R_3R_4)^2 + R_3{}^2\left(L_4\omega - \dfrac{1}{C_4\omega}\right)^2}$$

$$= \frac{M^2\omega^2 \,/\, R_3R_4}{\dfrac{1}{4}\left(\dfrac{M^2\omega^2}{R_3R_4} + 1\right)^2 + \dfrac{1}{4}\dfrac{L_4{}^2\omega^2}{R_4{}^2}\left\{1 - \left(\dfrac{\omega_4}{\omega}\right)^2\right\}^2}$$

$$= \frac{\tau^2 \,/\, \eta_3\eta_4}{\dfrac{1}{4}\left(\dfrac{\tau^2}{\eta_3\eta_4} + 1\right)^2 + \dfrac{1}{4\,\eta_4{}^2}\left\{1 - \left(\dfrac{\lambda}{\lambda_4}\right)^2\right\}^2}.$$

Equation (46) is to be employed in place of (41) whenever

$$\tau^2 \,/\, \eta_3\eta_4 \lesseqgtr 1.$$

An interesting case arises when $\tau^2 \,/\, \eta_3\eta_4 = 1$. Equation (46) then simplifies to

$$(47)\qquad \left(\frac{Y_{(U_{opt})}}{Y_{max\ max}}\right)^2 = 1 + \frac{1}{\dfrac{1}{4\,\eta_4{}^2}\left\{1 - \left(\dfrac{\lambda}{\lambda_4}\right)^2\right\}^2}.$$

Equation (47) holds only when $\tau^2 = \eta_4\eta_4$; that is, $M^2\omega^2 = R_4R_4$. In this case also $U_{opt} = V_{opt} = 0$.

Computation and Discussion of Case I. In the above paragraphs the equations have been derived for the current developed in Circuit

TABLE IV.

$$\text{Given } \tau = .30 \qquad \tau^2 = .09$$
$$\eta_3 = .01$$
$$\eta_4 = .01 \qquad \frac{\tau^2}{\eta_3\,\eta_4} = 900$$

$$\left(\frac{Y_{(U_{opt})}}{Y_{max\ max}}\right)^2 = \frac{1}{\frac{(k-1)^2}{4}\left(\frac{\tau^2}{\eta_3\eta_4} - 1\right) + 1} = \frac{1}{225(k-1)^2 + 1}$$

$$\left(\frac{\lambda_4}{\lambda}\right) = \frac{1}{1 \pm k\eta_4\sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}} = \frac{1}{1 \pm .3\,k}.$$

k.	λ_4/λ		$\left[\frac{Y}{Y_{mm}}\right]^2$	k.	λ_4/λ		$\left[\frac{Y}{Y_{mm}}\right]^2$
	With plus sign.	With minus sign.			With plus sign.	With minus sign.	
1.8	.806	1.490	.007	.95	.882	1.183	.640
1.7	.814	1.428	.009	.9	.887	1.170	.307
1.6	.822	1.387	.012	.8	.898	1.147	.100
1.5	.831	1.348	.018	.7	.909	1.125	.047
1.4	.839	1.313	.027	.6	.921	1.104	.027
1.3	.848	1.280	.047	.5	.933	1.084	.018
1.2	.857	1.250	.100	.4	.945	1.066	.012
1.1	.868	1.222	.307	.3	.958	1.048	.009
1.05	.872	1.208	.640	.2	.971	1.030	.007
1.0	.877	1.196	1.000				

IV for any adjustment of this circuit, while Circuit III is kept at its optimum value. This corresponds to keeping the antenna circuit of a wireless telegraph station set at its best value for given conditions and tuning the system by variation of Circuit IV. Assuming any given constants of the two circuits and any given wave length of incident waves, we can now make computations which will give the shape

and form of the resonance curve. In making the calculations if $M^2\omega^2 > R_3R_4$ we can take convenient values of the parameter k and calculate the wave-length adjustment corresponding to the given values of k (equation 44) and also the relative current for the same values of k (equation 41). The results will give relative current corresponding to various wave-length adjustments.

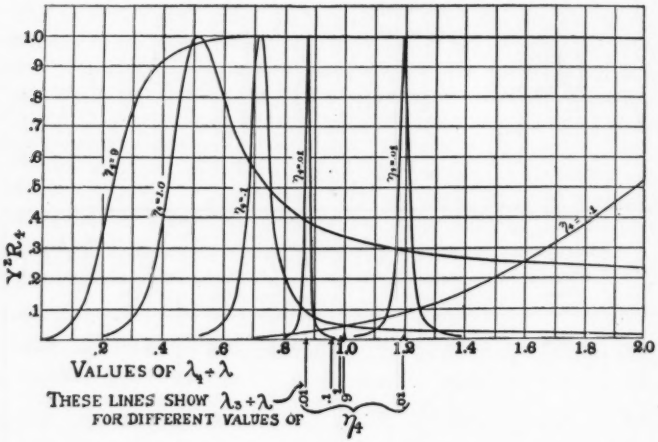

FIGURE 8. Resonance curves giving theoretical relation of current-square times resistance of Circuit IV to wave-length adjustment in the neighborhood of the optimum adjustment of Circuit IV when Circuit III is at optimum — for $\eta_3 = .01$, $\tau = .30$.

Case I with $\eta_4 = .01$, $\eta_3 = .01$, $\tau = .30$. — A sample set of computed results assuming $\tau = .30$, $\eta_3 = .01$ and $\eta_4 = .01$ is given in Table IV.

These results are plotted as the two curves marked "$\eta_4 = .01$" of Figure 8. The abscissas are values of the wave-length adjustment relative to the wave length of the incident waves (λ_4/λ). The ordinates are values of $\left(\frac{Y}{Y_{max\ max}}\right)^2$ but since $Y_{max\ max} = \frac{E}{2\sqrt{R_3R_4}}$, and since all the curves of the figure are plotted with constant R_3 (i.e., constant η_3), constant E, and constant M, it is perhaps more instructive to regard the ordinates of Figure 8 as relative values of Y^2R_4, and they are so designated in the figure. Referring again to the figure, these two curves marked "$\eta_4 = .01$" are obtained as resonance curves

of square of current in Circuit IV plotted against wave-length adjustment of Circuit IV, and are obtained by setting Circuit III at its best value and leaving it at that value during the tuning. The appropriate best-value settings of the wave length of Circuit III are indicated by the positions of the two lines marked "$\eta_4 = .01$" at the lower margin of the figure. The short-wave adjustment of λ_3 (at $\lambda_3/\lambda = .877$) is required for the short-wave resonance curve (with its maximum at $\lambda_4/\lambda = .877$), and the long-wave adjustment of λ_3 (at $\lambda_3/\lambda = 1.196$) is required for the long-wave resonance curve (with maximum at $\lambda_4/\lambda = 1.196$). It is seen that in this particular case, with $\eta_3 = \eta_4 = .01$, the resonant adjustment of Circuit III and that of Circuit IV have the same wave-length $\lambda_3 = \lambda_4$; and because of the smallness of the damping factors η_3 and η_4, the two curves are sharp.

Case I (continued). Assuming again $\tau = .30$, $\eta_3 = .01$, *but with* $\eta_4 = .1$. — Suppose now that we give to Circuit IV a higher resistance so that $\eta_4 = .1$. This may be done by using a higher resistance detector in Circuit IV. This will cut down the maximum value of the current in Circuit IV, but will leave the square of the current times the resistance (namely, Y^2R_4) the same as before, so that the curve in terms of Y^2R_4 will have the same maximum amplitude for $\eta_4 = .1$ as for $\eta_4 = .01$. Complete computations from equations (44) and (41) show that the curve will have the form and position given in Figure 8 and marked "$\eta_4 = .1$." The corresponding appropriate adjustment of Circuit III is given by the line marked "$\eta_4 = .1$" of the lower margin. The curve going out to the right, also marked "$\eta_4 = .1$" in Figure 8, is a part of another possible resonance curve in this case. This second resonance curve culminates beyond the limit of the figure with its maximum at $\lambda_4/\lambda = 4.18$, and requires the adjustment of λ_3/λ at 1.05.

The results of the computations in this case show in an interesting manner the necessity of tuning both circuits to get resonance, and show how markedly the adjustment of Circuit IV may be affected by the adjustment of Circuit III; since with the constants here assumed, the change of Circuit III from $\lambda_3/\lambda = .96$ to $\lambda_3/\lambda = 1.05$ necessitates the shifting of λ_4/λ from .716 to 4.18. The resonance in the former case is sharp, and that in the latter case is very dull.

Case I (continued). Assuming again $\tau = .30$, $\eta_3 = .01$, *while* η_4 *is made* $= 1$. We obtain the resonance curve marked $\eta_4 = 1.0$ (Figure 8) with appropriate adjustment of λ_3/λ_4 at .986. In this case the second resonance value in the region of long waves is imaginary.

Case I (continued). $\tau = .30$, $\eta_3 = .01$, *while* $\eta_4 = 9$. — This is the special case requiring the use of equation (47) and gives the curve

marked "$\eta_4 = 9$," requiring λ_3/λ to be 1.0. This curve is almost flat on top and for this condition tuning with Circuit IV is impossible. There is left here, as with all the curves of this figure, the alternative of leaving Circuit IV fixed at its best value and tuning with Circuit III. This is Case II, which we presently come to consider. It is proposed first, however, to present a new set of curves under Case I with a

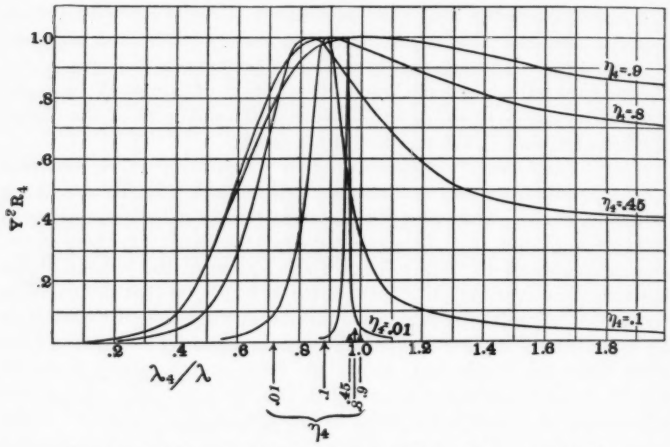

FIGURE 9. Curves similar to Figure 8, but with $\eta_3 = .1$, $\tau = .30$.

larger damping factor (η_3) in Circuit III, corresponding to the antenna circuit.

Case I (continued). Assuming $\tau = .30$, $\eta_3 = .1$, *and* η_4 *various.* — The curves for these conditions are plotted in Figure 9 for only the short-wave adjustment. The corresponding long-wave adjustments are presented in Figure 10.

The Equations and Computation of Case II. — It is now proposed to set Circuit IV at its best value, with any given constants of circuits, and to tune with Circuit III. The equations giving the current in Circuit IV for various values of wave length of Circuit III, are obtained from (41) and (44) by a simple interchange of subscripts, — giving

$$\left(\frac{Y_{(v_{opt})}}{Y_{max\ max}}\right)^2 = \frac{1}{\frac{(k-1)^2}{4}\left(\frac{\tau^2}{\eta_3\eta_4} - 1\right) + 1}, \tag{48}$$

where k is determined by the equation

$$\left(\frac{\lambda_3}{\lambda}\right)^2 = \frac{1}{1 \pm k\eta_3 \sqrt{\frac{\tau^2}{\eta_3\eta_4} - 1}} \tag{49}$$

By a comparison of (48) and (49) with equations (41) and (44) it will be seen that the method of Case II will give sharper resonance

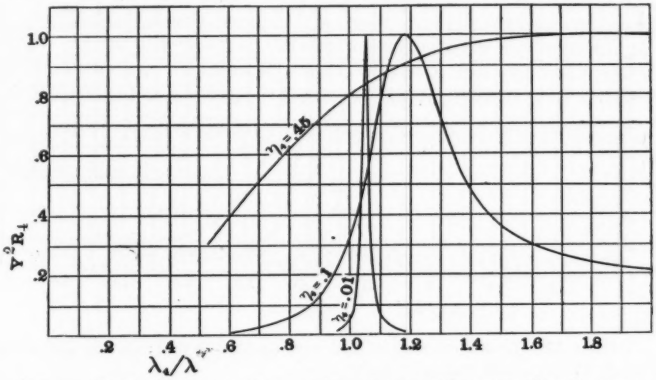

FIGURE 10. Long-wave adjustments corresponding to the short-wave adjustments of Figure 9.

than that of Case I whenever $\eta_3 < \eta_4$. With the high resistance detectors in ordinary use in wireless telegraphy, η_3 is generally much less than η_4, and the sharp tuning is best attained by the method of Case II; that is, by fixing the condenser circuit (Circuit IV) by successive approximations to a setting somewhere near its best value for the given incident waves, and then making the final adjustment by changing Circuit III.

Three numerical examples are given in the curves of Figure 11. For all these curves τ is assumed to be .30, and η_4 is taken as 1.0. The values of η_3 are marked on the respective curves.

Application to Actual Wireless Telegraphic Cases. — Now the corresponding actual wireless-telegraph resonance curves taken with coupled circuits, one of which has distributed capacity, will be less sharp than the corresponding curves here computed and drawn, so that as soon as we know the damping constants and the coefficient of coupling of the wireless-telegraph circuits we can select from the curves here computed

a set of curves that will be more selective than the wireless-telegraph curves, and we can in this way fix a limit to the sharpness of resonance that can be attained in practice. Some computations of this character on the extent to which interference can be prevented under certain conditions, assumed as practical, have been published elsewhere.[4]

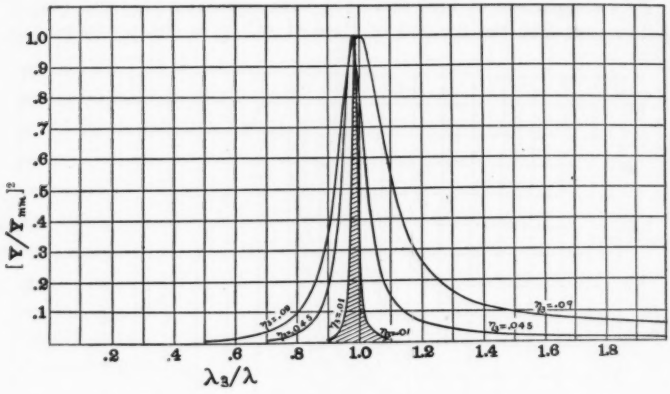

FIGURE 11. Resonance curves obtained by tuning with Circuit III while Circuit IV is at best value. $\tau = .30$, $\eta_4 = 1$.

VIII. THE MAXIMUM CURRENT, AND DETECTOR RESISTANCE.

Equation (23) is an expression for the maximum current that can be obtained in Circuit IV, when the inductances and capacities of Circuits III and IV are given their best values.

An interesting fact that can be obtained from an examination of equation (23) is that the square of the current multiplied by R_4 gives a quantity independent of R_4. This means that the heat developed in Circuit IV is independent of R_4; that is to say, the same amount of heat is developed in Circuit IV, at best resonance, whether a high-resistance or a low-resistance detector is employed. This means that, if the detector is an instrument for measuring heat, a low-resistance detector would be as sensitive as a high-resistance detector if it were not for the fact that the lower the resistance of the detector the less the proportion of the total heat that is developed in the detector itself,

[4] Pierce, Principles of Wireless Telegraphy, McGraw-Hill Book Company, New York, 1910.

because of the larger development of heat in the rest of the Circuit IV. Since the resonance is sharper with the low-resistance circuit, the resistance of a thermal detector, provided its indications are proportional merely to the heat developed in it, ought to be as low as is consistent with the localization of a large part of the energy in the detector; that is, for example, in order to get 9/10 of the maximum effect, the resistance of the detector, if its indications are proportional merely to the heat developed in it, ought to be nine times the high-frequency resistance of the rest of Circuit IV.

Similar considerations apply to a detector of the electrodynamometer type. If the deflections of the electrodynamometer are proportional to $n^2 Y^2$, where n is the number of turns of wire in the coil, and if the size of the channel of windings is fixed so that the resistance R of the detector is $\frac{\rho l}{S}$, l and S being the length and cross section of the wire in the coil, then we have

$$l = 2\pi r \cdot n,$$

in which r is the mean radius of the windings; and approximately

$$S = \frac{A}{n},$$

A being the area of the channel.

Therefore,
$$R = \frac{\rho\, 2\, \pi r n^2}{A},$$

or
$$R \sim n^2.$$

Whence if the deflection,

$$D \sim n^2 Y^2,$$
$$D \sim R\, Y^2.$$

In this, R is the resistance of the detector alone. Now according to equation (23) the quantity of $R_4 Y^2$ is not changed by changing R_4. Hence if the resistance R is made nine times the resistance of the rest of Circuit IV, the deflection of the high-frequency dynamometer will be 9/10 as large as would be obtained with a detector of very high resistance, and the resonance with the low resistance detector will be much sharper than with the detector of very high resistance.

However, it must be borne in mind that this conclusion holds only between different detectors of the same type, and presupposes that the factor by which $R Y^2$ is to be multiplied to get the deflection or other

indication is independent of R. In the case of the dynamometer or hot-wire ammeter the factor of convertibility of the energy of rapid alternations into deflection is probably fairly constant but is small; whereas, with certain other types of detectors, notably the electrolytic and the crystal rectifiers the convertability of the energy of rapid alternations into direct current energy is not constant, and appears to be relatively large only provided the resistance of the detector is large. This has constrained wireless telegraphic practice to high-resistance detectors, with the consequent deficiency in sharpness of resonance. The analysis given in the present paper shows that there is no inherent necessity in using these high-resistance detectors provided only detectors of lower resistance can be found with a large efficiency in converting electric energy of rapid alternations into energy of direct current or slowly-varying periodic current.

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October, 1910.